

# Transversals of Longest Paths and Cycles

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## Abstract

Let  $G$  be a graph of order  $n$ . Let  $\text{lpt}(G)$  be the minimum cardinality of a set  $X$  of vertices of  $G$  such that  $X$  intersects every longest path of  $G$  and define  $\text{lct}(G)$  analogously for cycles instead of paths. We prove that

- $\text{lpt}(G) \leq \left\lceil \frac{n}{4} - \frac{n^{2/3}}{90} \right\rceil$ , if  $G$  is connected,
- $\text{lct}(G) \leq \left\lceil \frac{n}{3} - \frac{n^{2/3}}{36} \right\rceil$ , if  $G$  is 2-connected, and
- $\text{lpt}(G) \leq 3$ , if  $G$  is a connected circular arc graph.

Our bound on  $\text{lct}(G)$  improves an earlier result of Thomassen and our bound for circular arc graphs relates to an earlier statement of Balister *et al.* the argument of which contains a gap. Furthermore, we prove upper bounds on  $\text{lpt}(G)$  for planar graphs and graphs of bounded tree-width.

**Keywords:** Longest path, longest cycle, transversal.

**MSC2010:** 05C38, 05C70

# 1 Introduction

It is well known that every two longest paths in a connected graph as well as every two longest cycles in a 2-connected graph intersect. While these observations are easy exercises, it is an open problem, originating from a question posed by Gallai [2], to determine the largest value of  $k$  such that for every connected graph and every  $k$  longest paths in that graph, there is a vertex that belongs to all of these  $k$  paths. The above remark along with examples constructed by Skupień [5] ensure that  $2 \leq k \leq 6$ .

We consider only simple, finite, and undirected graphs and use standard terminology. For a graph  $G$ , we define  $\mathcal{P}(G)$  to be the collection of all longest paths of  $G$  and a *longest path transversal* of  $G$  to be a set of vertices that intersects every longest path of  $G$ . Let  $\text{lpt}(G)$  be the minimum cardinality of a longest path transversal of  $G$ . We define  $\mathcal{C}(G)$ , a *longest cycle transversal*, and the parameter  $\text{lct}(G)$  analogously for cycles instead of paths.

The intersections of longest paths and cycles have been studied in detail and Zamfirescu [8] gave a short survey. In the present paper we prove upper bounds on  $\text{lpt}(G)$  and  $\text{lct}(G)$ . Our bound on  $\text{lct}(G)$  for a 2-connected graph  $G$  improves an earlier result of Thomassen [6]. Balister *et al.* [1] showed that for every connected interval graph, there is a vertex that belongs to every longest path. Furthermore, their work [1] contains the statement that for every connected circular arc graph, there is a vertex that belongs to every longest path. Unfortunately, we believe that the argument they provide has a gap. We shall explain the approach of Balister *et al.*, the problem with their argument, and give a proof of a weaker result, specifically that every connected circular arc graph contains a longest path transversal of order at most 3.

# 2 Results

We start by proving a lemma that allows us to exploit the structure of some particular matchings to find long paths and cycles.

**Lemma 1.** *If  $G = (P \cup Q) + M$  where  $P : u_1 \dots u_\tau$  and  $Q : v_1 \dots v_\tau$  are paths and  $M$  is a matching of edges between  $V(P)$  and  $V(Q)$  that has a partition  $M = M_1 \cup \dots \cup M_q$  such that*

(a)  *$|M_i|$  is either 1 or even for  $i \in [q]$  and*

(b) *if  $u_{i_1}v_{i_2} \in M_i$  and  $u_{j_1}v_{j_2} \in M_j$  for  $i, j \in [q]$ , then*

$$(j_1 - i_1)(j_2 - i_2) \begin{cases} < 0, & \text{if } i = j \text{ and} \\ > 0, & \text{if } i \neq j, \end{cases}$$

*that is, the edges in one of the sets  $M_i$  are pairwise “crossing” and the edges in distinct sets  $M_i$  are pairwise “parallel”,*

*then  $G$  contains a path between a vertex in  $\{u_1, v_1\}$  and a vertex in  $\{u_\tau, v_\tau\}$  of order at least  $\tau + |M|$ .*

*Proof.* If  $i_0 = 1$ ,  $i_{|M|+1} = \tau$ , and  $u_{i_1}, \dots, u_{i_{|M|}}$  with  $1 \leq i_1 < \dots < i_{|M|} \leq \tau$  are the vertices of  $P$  that are incident with edges in  $M$ , then a subpath of  $P$  of the form  $u_{i_j} \dots u_{i_{j+1}}$  with odd/even  $j$  is called an *odd/even segment* of  $P$ , respectively. Odd/even segments of  $Q$  are defined analogously.

The odd segments of  $P$ ,  $M$ , and the even segments of  $Q$  define a path  $P'$ . Similarly, the even segments of  $P$ ,  $M$ , and the odd segments of  $Q$  define a path  $Q'$ . See Figure 1 for an illustration. Since  $E(P') \cap E(Q') = M$  and  $E(P') \cup E(Q') = E(P) \cup E(Q) \cup M$ , the longer of the two paths satisfies the desired properties.  $\square$

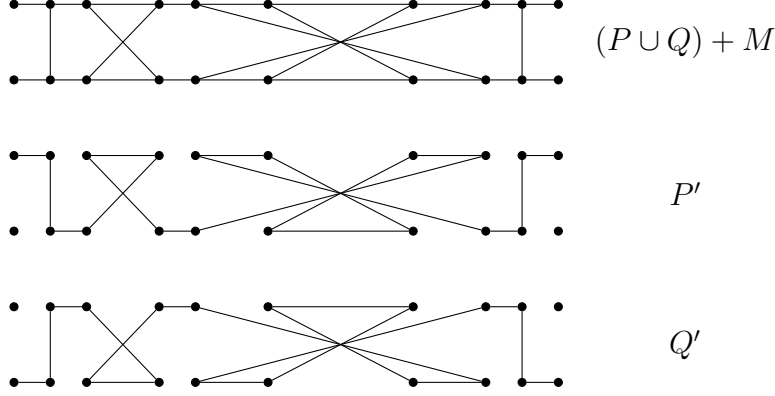


Figure 1: The two paths  $P'$  and  $Q'$  for a set  $M$  with  $|M_1| = |M_4| = 1$ ,  $|M_2| = 2$ , and  $|M_3| = 4$ .

We proceed to our first main result. Note that in the proof of Theorem 2, as well as of Theorem 3 below, we did not try to minimize the factor of  $n^{\frac{2}{3}}$ . The point of these two results is that  $\text{lpt}(G)$  is strictly less than  $n/4$  and  $\text{lct}(G)$  is strictly less than  $n/3$ , respectively.

**Theorem 2.** *If  $G$  is a connected graph of order  $n$ , then  $\text{lpt}(G) \leq \left\lceil \frac{n}{4} - \frac{n^{2/3}}{90} \right\rceil$ .*

*Proof.* Let  $G$  be a connected graph of order  $n$ . Let  $\epsilon = \frac{1}{90}n^{-\frac{1}{3}}$  and  $\tau = \left\lceil \left(\frac{1}{4} - \epsilon\right)n \right\rceil$ . For a contradiction, we assume that  $\text{lpt}(G) > \tau$ . Let  $P : u_1 \dots u_\ell$  be a longest path of  $G$ . Since  $V(P)$  as well as every set of  $n - \ell + 1$  vertices of  $G$  are longest path transversals, we obtain

$$\left(\frac{1}{4} - \epsilon\right)n \leq \tau < \ell < n - \tau + 1 \leq \left(\frac{3}{4} + \epsilon\right)n + 1. \quad (1)$$

Let  $p = \left\lceil \frac{\ell - \tau}{2} \right\rceil$ . Since the set  $T = \{u_i : p + 1 \leq i \leq p + \tau\}$  is too small to be a longest path transversal of  $G$ , there is a path  $P' : v_1 \dots v_\ell$  in  $G - T$ . Since  $G$  is connected, the paths  $P$  and  $P'$  intersect.

If  $V(P) \cap V(P') \subseteq \{u_1, \dots, u_p\}$ , then let  $v_x = u_r \in V(P) \cap V(P')$  be such that  $r$  is maximum. We may assume that  $x \geq \frac{\ell+1}{2}$ . Now  $v_1 \dots v_x u_{r+1} \dots u_\ell$  is a path of order at least  $x + \ell - p \geq \frac{\ell+1}{2} + \ell - \frac{\ell-\tau+1}{2} = \ell + \frac{\tau}{2} > \ell$ , which is a contradiction. Hence  $P$  and  $P'$  intersect in a vertex in  $\{u_1, \dots, u_p\}$  as well as a vertex in  $\{u_{p+\tau+1}, \dots, u_\ell\}$ . Let  $v_x = u_r$  be in  $V(P') \cap \{u_1, \dots, u_p\}$  such that  $r$  is maximum and  $v_y = u_s$  be in  $V(P') \cap \{u_{p+\tau+1}, \dots, u_\ell\}$  such that  $s$  is minimum. We may assume that  $x < y$ .

Since  $v_1 \dots v_x u_{r+1} \dots u_{s-1} v_y \dots v_\ell$  is a path of order at least  $\ell - (y - x - 1) + \tau$ , we obtain  $y - x - 1 \geq \tau$ . Since  $u_{s-1} \dots u_{r+1} v_x \dots v_\ell$  is a path of order at least  $\tau + \ell - (x - 1)$ , we obtain  $x - 1 \geq \tau$ . Since  $u_{r+1} \dots u_{s-1} v_y \dots v_1$  is a path of order at least  $\tau + y$ , we obtain  $\ell - y \geq \tau$ .

Choosing four vertex-disjoint paths  $A : a_1 \dots a_\tau$ ,  $B : b_1 \dots b_\tau$ ,  $C : c_1 \dots c_\tau$ , and  $D : d_1 \dots d_\tau$  as subpaths of the four paths  $P'[\{v_1, \dots, v_{x-1}\}]$ ,  $P[\{u_{r+1}, \dots, u_{s-1}\}]$ ,  $P'[\{v_{x+1}, \dots, v_{y-1}\}]$ , and  $P'[\{v_{y+1}, \dots, v_\ell\}]$ , respectively, we obtain the existence of two vertex-disjoint sets  $X$  and  $Y$  in

$V(G) \setminus (V(A) \cup V(B) \cup V(C) \cup V(D))$  with  $|X \cup Y| \leq n - 4\tau \leq 4\epsilon n$  such that  $X$  contains a path between some neighbors of any two of the vertices  $a_1, b_1$ , and  $c_1$ , and  $Y$  contains a path between some neighbors of any two of the vertices  $b_\tau, c_\tau$ , and  $d_1$ . See Figure 2 for an illustration.

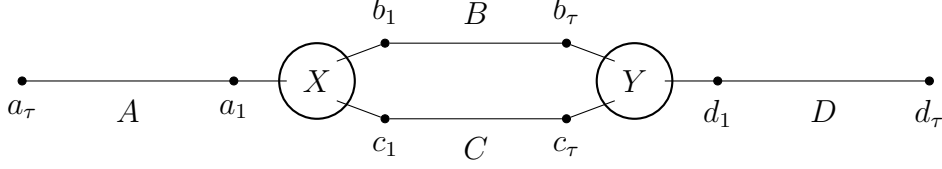


Figure 2: The paths  $A, B, C$ , and  $D$  and the two sets  $X$  and  $Y$ .

If  $a_i b_j$  is an edge of  $G$  with  $j \geq i$ , then the path  $a_\tau \dots a_i b_j \dots b_1$ , a path in  $X$  between neighbors of  $b_1$  and  $c_1$ , the path  $C$ , a path in  $Y$  between neighbors of  $c_\tau$  and  $d_1$ , and the path  $D$  form a path of order at least  $3\tau + (j - i + 1) + 2$ . By (1), this implies that

$$j - i \leq \lceil 4\epsilon n \rceil - 3. \quad (2)$$

Our goal is now to prove the existence of a vertex cover  $T_{A,B}$  of small order for the bipartite graph  $G_{A,B}$  with bipartition  $V(A)$  and  $V(B)$  formed by the edges between these two sets. First, note that if  $4\epsilon n \leq 2$ , then (2) implies that this bipartite graph is edgeless, so it is enough to set  $T_{A,B} = \emptyset$ .

Assume now that  $4\epsilon n > 2$ . Let  $N$  be a maximum matching of  $G_{A,B}$ . Let  $I = \left\lceil \left\lceil \frac{\tau}{2(\lceil 4\epsilon n \rceil - 3)} \right\rceil \right\rceil$ . For  $i \in I$ , let  $N_i$  be the set of edges in  $N$  that are incident with a vertex in

$$\{a_j : 2(\lceil 4\epsilon n \rceil - 3)(i - 1) + 1 \leq j \leq \min\{\tau, 2(\lceil 4\epsilon n \rceil - 3)i\}\}.$$

By (2), if  $a_{i_1} b_{i_2} \in N_i$  and  $a_{j_1} b_{j_2} \in N_j$  with  $j - i \geq 2$ , then  $(j_1 - i_1)(j_2 - i_2) > 0$ , that is, the two edges are parallel in the sense of Lemma 1. Without loss of generality, we may assume that  $\bigcup_{i \in I : i \text{ odd}} N_i$  contains at least half the edges of  $N$ . Since  $N$  is a matching,  $|N_i| \leq 2(\lceil 4\epsilon n \rceil - 3)$  for every  $i \in I$ . Since permutation graphs are perfect [3, Chapter 7], each  $N_i$  contains a set of at least  $\sqrt{|N_i|} \geq \frac{|N_i|}{\sqrt{2(\lceil 4\epsilon n \rceil - 3)}}$  edges that are either all pairwise parallel or all pairwise crossing in the sense of Lemma 1. This implies that  $N$  contains a subset  $M_0$  that satisfies condition (b) from Lemma 1 with  $|M_0| \geq \sum_{i \in I : i \text{ odd}} \frac{|N_i|}{\sqrt{2(\lceil 4\epsilon n \rceil - 3)}} \geq \frac{1}{2\sqrt{2(\lceil 4\epsilon n \rceil - 3)}} |N|$ . By removing a set of at most  $|M_0|/3$  edges from  $M_0$ , we obtain a matching  $M$  that satisfies both conditions from Lemma 1 with  $|M| \geq \frac{2}{3} |M_0| \geq \frac{1}{3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}} |N|$ .

If  $N$  has order at least  $(4\epsilon n - 1)3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}$ , then  $N$  contains a matching  $M$  as in Lemma 1 of order at least  $4\epsilon n - 1$ . Thus by Lemma 1, the graph  $(A \cup B) + M$  contains a path  $Q$  between  $\{a_1, b_1\}$  and  $\{a_\tau, b_\tau\}$  of order at least  $\tau + |M|$ . Now the path  $Q$ , a path in  $X$  between neighbors of a vertex in  $\{a_1, b_1\}$  and  $c_1$ , the path  $C$ , a path in  $Y$  between neighbors of  $c_\tau$  and  $d_1$ , and the path  $D$  form a path of order at least  $3\tau + |M| + 2 \geq \left(\frac{3}{4} + \epsilon\right)n + 1$ , which contradicts (1). Hence the bipartite graph  $G_{A,B}$  has no matching of order at least  $(4\epsilon n - 1)3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}$ . Now König's theorem [4] implies that  $G_{A,B}$  has a vertex cover  $T_{A,B}$  of order less than  $(4\epsilon n - 1)3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}$ .

Similar arguments yield that for every two distinct paths  $Q, R \in \{A, B, C, D\}$ , the bipartite graph  $G_{Q,R}$  with bipartition  $V(Q)$  and  $V(R)$  formed by the edges between these two sets has a vertex cover  $T_{Q,R}$  of order less than  $(4\epsilon n - 1)3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}$  if  $4\epsilon n > 2$  and of order 0 otherwise. (If, for instance,  $a_i d_j$  is an edge of  $G$  with  $j \geq i$ , then the path  $a_\tau \dots a_i d_j \dots d_1$ , a path in  $Y$  between neighbors of  $d_1$  and  $b_\tau$ , the path  $B$ , a path in  $X$  between neighbors of  $b_1$  and  $c_1$ , and the path  $C$  again form a path of order at least  $3\tau + (j - i + 1) + 2$  and we can argue as above.)

Let  $T' = X \cup Y \cup \bigcup_{\{Q,R\} \in \binom{\{A,B,C,D\}}{2}} T_{Q,R}$ . Since every component of  $G - T'$  has order at most  $\tau$ , the set  $T'$  is a longest path transversal of  $G$  of order less than  $4\epsilon n + 6(4\epsilon n - 1)3\sqrt{2(\lceil 4\epsilon n \rceil - 3)}$  if  $4\epsilon n > 2$  and of order at most  $4\epsilon n$  otherwise. For  $\epsilon = \frac{1}{90}n^{-\frac{1}{3}}$ , it follows that  $|T'| \leq \left(\frac{1}{4} - \epsilon\right)n$ , which yields the final contradiction.  $\square$

For the fractional version of the longest path transversal problem, a much stronger result is possible. In fact, for every connected graph  $G$ , there is a function  $t: V(G) \rightarrow [0, 1]$  such that

$$\begin{aligned} \sum_{u \in V(G)} t(u) &\leq \sqrt{n} \quad \text{and} \\ \sum_{u \in V(P)} t(u) &\geq 1 \quad \text{for every } P \in \mathcal{P}(G). \end{aligned}$$

Indeed, if the largest order of the paths in  $\mathcal{P}(G)$  is at most  $\sqrt{n}$ , then let  $t$  be the characteristic function of  $V(P)$  for some  $P \in \mathcal{P}(G)$ , otherwise let  $t$  be the constant function of value  $\frac{1}{\sqrt{n}}$ .

Confirming a conjecture by Zamfirescu [7], Thomassen [6] proved that  $\text{lct}(G) \leq \left\lceil \frac{|V(G)|}{3} \right\rceil$  for every graph  $G$ , which is best possible for the class of connected graphs in view of a disjoint union of cycles of length 3 to which bridges are added. For 2-connected graphs, though, this bound can be improved as follows.

**Theorem 3.** *If  $G$  is a 2-connected graph of order  $n$ , then  $\text{lct}(G) \leq \left\lceil \frac{n}{3} - \frac{n^{2/3}}{36} \right\rceil$ .*

*Proof.* Let  $G$  be a 2-connected graph of order  $n$ . Let  $\epsilon = \frac{1}{36}n^{-\frac{1}{3}}$  and  $\tau = \left\lceil \left(\frac{1}{3} - \epsilon\right)n \right\rceil$ . For a contradiction, we assume that  $\text{lct}(G) > \tau$ . Let  $C: u_0 \dots u_{\ell-1} u_0$  be a longest cycle of  $G$ . Since  $V(C)$  as well as every set of  $n - \ell + 1$  vertices of  $G$  are longest cycle transversals, we obtain

$$\left(\frac{1}{3} - \epsilon\right)n \leq \tau < \ell < n - \tau + 1 \leq \left(\frac{2}{3} + \epsilon\right)n + 1. \quad (3)$$

Since the set  $T = \{u_0, \dots, u_{\tau-1}\}$  is too small to be a longest cycle transversal of  $G$ , there is a cycle  $C': v_0 \dots v_{\ell-1} v_0$  in  $G - T$ . Since  $G$  is 2-connected, the cycles  $C$  and  $C'$  intersect in at least two vertices. We may assume that  $v_0 = u_r$  is the first and  $v_k = u_s$  is the last common vertex of  $C$  and  $C'$  following the path  $C - T$  from  $u_\tau$  to  $u_{\ell-1}$ , that is  $r < s$ .

Since  $v_0 \dots v_k u_{s+1} \dots u_{\ell-1} u_0 \dots u_{r-1}$  is a cycle of length at least  $k+1+\tau$ , we obtain  $\ell - k - 1 \geq \tau$ . Since  $v_{k+1} \dots v_{\ell-1} u_r \dots u_0 u_{\ell-1} \dots u_s$  is a cycle of length at least  $\ell - (k - 1) + \tau$ , we obtain  $k - 1 \geq \tau$ .

Choosing three vertex-disjoint paths  $P: x_1 \dots x_\tau$ ,  $Q: y_1 \dots y_\tau$ , and  $R: z_1 \dots z_\tau$  as subpaths of the three paths  $C[T]$ ,  $C'[\{v_1, \dots, v_{k-1}\}]$ , and  $C'[\{v_{k+1}, \dots, v_{\ell-1}\}]$ , respectively, we obtain the existence of two vertex-disjoint sets  $X$  and  $Y$  in  $V(G) \setminus (V(P) \cup V(Q) \cup V(R))$  with  $|X \cup Y| \leq n - 3\tau \leq 3\epsilon n$  such that  $X$  contains a path between some neighbors of every two of

the vertices  $x_1, y_1$ , and  $z_1$ , and  $Y$  contains a path between some neighbors of every two of the vertices  $x_\tau, y_\tau$ , and  $z_\tau$ . See Figure 3 for an illustration.

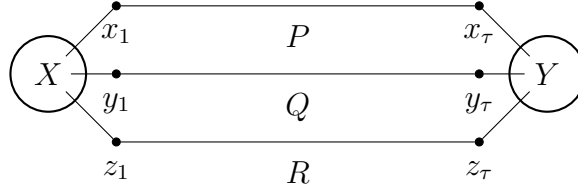


Figure 3: The paths  $P$ ,  $Q$ , and  $R$  and the two sets  $X$  and  $Y$ .

If  $x_i y_j$  is an edge of  $G$  with  $j \geq i$ , then the path  $y_1 \dots y_j x_i \dots x_\tau$ , a path in  $Y$  between neighbors of  $x_\tau$  and  $z_\tau$  the path  $R$ , and a path in  $X$  between neighbors of  $y_1$  and  $z_1$  form a cycle of length at least  $2\tau + (j - i + 1) + 2$ . By (3), this implies that  $j - i \leq \lceil 3\epsilon n \rceil - 3$ . Using Lemma 1 as in the proof of Theorem 2, we infer that for every two distinct  $A, B \in \{P, Q, R\}$ , the bipartite graph  $G_{A,B}$  with bipartition  $V(A)$  and  $V(B)$  formed by the edges between these two sets has a vertex cover  $T_{A,B}$  of order less than  $(3\epsilon n - 1)3\sqrt{2(\lceil 3\epsilon n \rceil - 3)}$  if  $3\epsilon n > 2$  and of order 0 otherwise. Since every component of  $G - (X \cup Y \cup T_{P,Q} \cup T_{P,R} \cup T_{Q,R})$  has order at most  $\tau$ , the set  $T' = X \cup Y \cup T_{P,Q} \cup T_{P,R} \cup T_{Q,R}$  is a longest cycle transversal of  $G$  of order less than  $3\epsilon n + 3(3\epsilon n - 1)3\sqrt{2(\lceil 3\epsilon n \rceil - 3)}$  if  $3\epsilon n > 2$  and of order at most  $3\epsilon n$  otherwise. For  $\epsilon = \frac{1}{36}n^{-\frac{1}{3}}$ , it follows that  $|T'| \leq \left(\frac{1}{3} - \epsilon\right)n$ , which yields the final contradiction.  $\square$

Since every two longest paths of a connected graph  $G$  intersect, it follows that  $\text{lpt}(G) \leq \left\lceil \frac{|\mathcal{P}(G)|}{2} \right\rceil$ . Similarly, if every  $k$  longest paths of a connected graph  $G$  would intersect for some  $k \geq 3$ , then it would follow that  $\text{lpt}(G) \leq \left\lceil \frac{|\mathcal{P}(G)|}{k} \right\rceil$ . The next result shows how to decrease the multiplicative constant  $1/2$  in the former bound at the cost of adding a square-root proportion of the total number of vertices in the graph.

**Proposition 4.** *If  $G$  is a connected graph and  $\alpha \geq 2$ , then*

$$\text{lpt}(G) \leq \frac{|\mathcal{P}(G)|}{\alpha} + \sqrt{\alpha |V(G)|}.$$

*Proof.* We proceed by induction on the order  $n$  of  $G$ , the statement being true if  $n = 1$ . Let  $n \geq 2$  and assume that the statement holds for all connected graphs of order less than  $n$ . Let  $G$  be a connected graph of order  $n$  and let  $\ell$  be the order of the longest paths in  $G$ .

We may assume that  $|\mathcal{P}(G)| > \sqrt{\alpha n}$  since otherwise we obtain a longest path transversal of the desired size by picking one vertex in each longest path of  $G$ . Next, since the vertex set of a longest path in  $G$  is a longest path transversal, we may also assume that  $\ell > \sqrt{\alpha n}$ .

For a vertex  $v \in V(G)$ , let  $p_v$  be the number of paths in  $\mathcal{P}(G)$  that contain  $v$ . We may assume that  $p_v < \alpha$  for every vertex  $v \in V(G)$ . Indeed, suppose that  $v$  is a vertex such that  $p_v \geq \alpha$ . In particular,  $p_v \geq 1$ . If the set  $\{v\}$  is a longest path transversal of  $G$ , then  $G$  satisfies the desired property. Otherwise let  $G' = G - v$  and note that  $G'$  contains a path of order  $\ell$ . Furthermore,  $|\mathcal{P}(G')| = |\mathcal{P}(G)| - p_v \leq |\mathcal{P}(G)| - \alpha$ . Note that all paths of order  $\ell$  in  $G'$  must belong to the same component of  $G'$ , since every two longest paths intersect. Let  $C$  be this component; thus  $\mathcal{P}(C) = \mathcal{P}(G')$ . The induction hypothesis applied to  $C$  yields that  $\text{lpt}(G') \leq \frac{|\mathcal{P}(G)|}{\alpha} - 1 + \sqrt{\alpha(n-1)}$ . As  $\text{lpt}(G) \leq \text{lpt}(G') + 1$ , we deduce that  $\text{lpt}(G) \leq \frac{|\mathcal{P}(G)|}{\alpha} + \sqrt{\alpha n}$ .

We now consider the number  $N$  of pairs  $(v, P)$  such that  $P \in \mathcal{P}(G)$  and  $v \in V(P)$ . On the one hand, since  $N = \sum_{v \in V(G)} p_v$ , we deduce from the previous observations that  $N < \alpha n$ . On the other hand, since  $N = \sum_{P \in \mathcal{P}(G)} |V(P)| = \ell |\mathcal{P}(G)|$ , the previous observations also imply that  $N > \alpha n$ . This contradiction concludes the proof.  $\square$

The minimum sizes of transversals of longest paths can be bounded in classes of graphs with small separators, such as planar graphs and graphs of bounded tree-width. As before, no effort is made to minimize the constant multiplicative factors appearing in the next two results.

**Proposition 5.** *If  $G$  is a connected planar graph of order at least 2, then*

$$\text{lpt}(G) \leq 9\sqrt{|V(G)|} \log |V(G)|.$$

*Proof.* We proceed by induction on the order  $n$  of  $G$ , the result being true if  $n = 2$ . Let  $n \geq 3$  and assume that the statement holds for all connected planar graphs of order at least 2 and less than  $n$ . Let  $G$  be a connected planar graph of order  $n$  and let  $\ell$  be the order of the longest paths in  $G$ . In particular,  $\ell \geq 2$ . Since  $G$  is planar, the separator theorem of Lipton and Tarjan ensures that  $G$  contains a set  $X$  of order at most  $2\sqrt{2}\sqrt{n}$  such that every component of  $G - X$  has order at most  $2n/3$ .

If  $X$  is a longest path transversal of  $G$ , then  $G$  satisfies the desired property. Otherwise,  $G - X$  contains a path of order  $\ell$ . Note that, since every two longest paths of  $G$  intersect, all paths of order  $\ell$  in  $G - X$  must be contained in the same component of  $G - X$ , which we call  $C$ . Moreover, the order of  $C$  is at most  $2n/3$  and at least  $\ell$ , so the induction hypothesis implies that  $C$  has a longest path transversal  $X'$  of order at most  $9\sqrt{2n/3} \log(2n/3)$ . Therefore,  $X \cup X'$  is a longest path transversal of  $G$  of order at most

$$\begin{aligned} 2\sqrt{2}\sqrt{n} + 9\sqrt{2n/3} \log(2n/3) &= 9\sqrt{2n/3} \log n + \sqrt{n} \cdot \left(2\sqrt{2} - 9\sqrt{2/3} \log(3/2)\right) \\ &\leq 9\sqrt{n} \log n \end{aligned}$$

since  $9\sqrt{2/3} \log(3/2) > 2\sqrt{2}$ . This concludes the proof.  $\square$

An analogous statement is true for graphs of bounded tree-width. Indeed, if  $G$  is a graph with tree-width at most  $k$ , then there is a set  $X$  of vertices of  $G$  of order at most  $k + 1$  such that every component of  $G - X$  has order at most  $|V(G)|/2$ . Consequently, an inductive reasoning similar to that made in the proof of Proposition 5 yields the following statement.

**Proposition 6.** *If  $G$  is a connected graph of tree-width at most  $k$  and order at least 2, then*

$$\text{lpt}(G) \leq 3k \log |V(G)|.$$

We proceed to circular arc graphs. We explain the approach of Balister *et al.* [1], the problem with their argument, and prove the following weaker result.

**Theorem 7.** *Let  $G$  be a circular-arc graph.*

*If  $G$  is connected, then  $\text{lpt}(G) \leq 3$ , and if  $G$  is 2-connected, then  $\text{lct}(G) \leq 3$ .*

Let  $G$  be a connected circular arc graph. Let  $C$  be a circle and let  $\mathcal{F}$  be a collection of open arcs of  $C$  such that  $G$  is the intersection graph of  $\mathcal{F}$ . In view of the result for interval graphs mentioned in the introduction, we may assume that  $C \subseteq \bigcup_{A \in \mathcal{F}} A$ . Furthermore, we may assume that all endpoints of arcs in  $\mathcal{F}$  are distinct.

Balister *et al.* [1] consider a collection  $\mathcal{K} = \{K_0, \dots, K_{n-1}\}$  of arcs in  $\mathcal{F}$  such that

- (0)  $C \subseteq \bigcup_{A \in \mathcal{K}} A$ ,
- (1)  $n$  is minimal, and
- (2) each  $K_i$  is maximal, that is, no arc in  $\mathcal{F}$  properly contains an arc in  $\mathcal{K}$ .

They may assume that  $n \geq 2$ , because otherwise,  $G$  has a universal vertex that belongs to every longest path or cycle. We consider the indices of the arcs in  $\mathcal{K}$  as elements of  $\mathbf{Z}_n$ , that is, modulo  $n$ .

A chain of order  $\ell$  in  $\mathcal{F}$  is a sequence  $\mathcal{P} : A_1 \dots A_\ell$  of distinct arcs in  $\mathcal{F}$  such that  $A_i \cap A_{i+1} \neq \emptyset$  for  $i \in [\ell - 1]$ . The chain  $\mathcal{P}$  is *closed*, if  $A_\ell \cap A_1 \neq \emptyset$ . Thus chains and closed chains in  $\mathcal{F}$  correspond to paths and cycles in  $G$ . For a chain  $\mathcal{P} : A_1 \dots A_\ell$  in  $\mathcal{F}$ , let  $\mathcal{K}(\mathcal{P}) = \{A_1, \dots, A_\ell\} \cap \mathcal{K}$ .

If  $\mathcal{P} : A_1 \dots A_\ell$  is a chain in  $\mathcal{F}$  of largest order, then Balister *et al.* [1, Lemma 3.1] proved that  $\mathcal{K}(\mathcal{P})$  is of the form  $\{K_i : i \in I\}$  for some contiguous and non-empty subset  $I$  of  $\mathbf{Z}_n$ . Their argument actually yields the same statement for closed chains, that is, if  $\mathcal{C}$  is a closed chain in  $\mathcal{F}$  of largest order, then  $\mathcal{K}(\mathcal{C})$  is of the form  $\{K_i : i \in J\}$  for some contiguous and non-empty subset  $J$  of  $\mathbf{Z}_n$ .

In the proof of their main result [1, Theorem 3.3] — stating that  $\text{lpt}(G) = 1$  — Balister *et al.* choose a chain  $\mathcal{P}$  in  $\mathcal{F}$  of largest order such that  $\mathcal{K}(\mathcal{P})$  has minimum order. They let  $\mathcal{K}(\mathcal{P})$  be  $\{K_{a+1}, \dots, K_{b-1}\}$  and assert that  $K_{b-1}$  belongs to  $\mathcal{K}(\mathcal{Q})$  for every chain  $\mathcal{Q}$  in  $\mathcal{F}$  of largest order, that is, the vertex of  $G$  corresponding to the arc  $K_{b-1}$  would belong to every longest path of  $G$ .

For a contradiction, they consider a chain  $\mathcal{Q}$  in  $\mathcal{F}$  of largest order such that  $K_{b-1} \notin \mathcal{K}(\mathcal{Q})$ . They set  $\mathcal{K}(\mathcal{Q}) = \{K_{\ell+1}, \dots, K_{m-1}\}$ . They deduce from the choice of  $\mathcal{P}$  that  $K_{\ell+1} \in \mathcal{K}(\mathcal{Q}) \setminus \mathcal{K}(\mathcal{P})$  since  $K_{b-1} \in \mathcal{K}(\mathcal{P}) \setminus \mathcal{K}(\mathcal{Q})$ . Using their Lemma 3.2 [1], they reorder the arcs in the chains  $\mathcal{P}$  and  $\mathcal{Q}$  and obtain chains  $\mathcal{P}^*$  and  $\mathcal{Q}^*$  containing the same arcs as  $\mathcal{P}$  and  $\mathcal{Q}$  in a possibly different order, respectively. They split these chains at  $K_{b-1}$  and  $K_{\ell+1}$  writing them as  $\mathcal{P}^* : \mathcal{P}_1 K_{b-1} \mathcal{P}_2$  and  $\mathcal{Q}^* : \mathcal{Q}_1 K_{\ell+1} \mathcal{Q}_2$ , respectively.

Their core statement is that  $\mathcal{C}_1 : \mathcal{P}_1 K_{b-1} \mathcal{R} K_{\ell+1} \mathcal{Q}_1^r$  and  $\mathcal{C}_2 : \mathcal{P}_2^r K_{b-1} \mathcal{R} K_{\ell+1} \mathcal{Q}_2$  are chains that satisfy the inequality  $|\mathcal{C}_1| + |\mathcal{C}_2| \geq 2 + |\mathcal{P}| + |\mathcal{Q}|$ , where  $\mathcal{R}$  is the possibly empty chain  $K_b \dots K_\ell$  and the exponent “ $r$ ” means reversal. In order to prove this statement, they have to show that no arc appears twice in these sequences. They give details only for  $\mathcal{C}_1$ . Their argument that  $\mathcal{C}_1$  is a chain heavily relies on the properties of the reordered chains  $\mathcal{P}^*$  and  $\mathcal{Q}^*$  guaranteed by their Lemma 3.2. In the proof of Lemma 3.2 these properties are established by iteratively shifting within  $\mathcal{P}$  the arc  $K_{b-1}$  to the beginning of  $\mathcal{P}$  and, similarly, by iteratively shifting within  $\mathcal{Q}$  the arc  $K_{\ell+1}$  to the beginning of  $\mathcal{Q}$ . After proving that  $\mathcal{C}_1$  is indeed a chain, they say that the same type of argument shows that  $\mathcal{C}_2$  is a chain as well.

This is the gap in their argument.

In order to use the same type of argument for  $\mathcal{C}_2$ , they would need reversed versions of the properties guaranteed by Lemma 3.2, that is, in order to establish these properties they would have to iteratively shift within  $\mathcal{P}$  the arc  $K_{b-1}$  to the end of  $\mathcal{P}$  and, similarly, to iteratively shift within  $\mathcal{Q}$  the arc  $K_{\ell+1}$  to the end of  $\mathcal{Q}$ . This may easily result in reorderings that are distinct



from  $\mathcal{P}^*$  and  $\mathcal{Q}^*$ . In view of this asymmetry, the suitably adapted chain  $\mathcal{C}_2$ , which would use the different reorderings of  $\mathcal{P}$  and  $\mathcal{Q}$ , need not satisfy the crucial inequality  $|\mathcal{C}_1| + |\mathcal{C}_2| \geq 2 + |\mathcal{P}| + |\mathcal{Q}|$  and the argument breaks down.

We proceed to the proof of our Theorem 7.

*Proof of Theorem 7.* Let  $G$  be a connected circular arc graph. We choose  $C$ ,  $\mathcal{F}$ , and  $\mathcal{K}$  exactly as above and we start by proving the following statement.

**Assertion.** *If  $\mathcal{P}$  and  $\mathcal{Q}$  are chains of largest order in  $\mathcal{F}$  such that*

$$\begin{aligned}\mathcal{K}(\mathcal{P}) &= \{K_{a+1}, \dots, K_{b-1}\} = \{K_i : i \in I(\mathcal{P})\} \quad \text{and} \\ \mathcal{K}(\mathcal{Q}) &= \{K_{\ell+1}, \dots, K_{m-1}\} = \{K_i : i \in I(\mathcal{Q})\}\end{aligned}$$

*are disjoint, then  $a = m - 1$  or  $b = \ell + 1$ , that is, the subsets  $I(\mathcal{P})$  and  $I(\mathcal{Q})$  of  $\mathcal{Z}_n$  are contiguous.*

To establish this assertion, assume on the contrary that  $a \neq m - 1$  and  $b \neq \ell + 1$ . Select a set  $S(\mathcal{P})$  of points of  $C$  such that  $S(\mathcal{P})$  contains a point in the intersection of every two consecutive arcs of  $\mathcal{P}$ . Define  $S(\mathcal{Q})$  analogously. If  $K_a$  or  $K_b$  would intersect  $S(\mathcal{P})$  or  $S(\mathcal{Q})$ , then  $K_a$  or  $K_b$  could be inserted into  $\mathcal{P}$  or  $\mathcal{Q}$ , respectively, contradicting the assumption that these chains are of largest order. If  $S(\mathcal{P})$  or  $S(\mathcal{Q})$  would intersect both arcs of  $C \setminus (K_a \cup K_b)$ , then some arc of  $\mathcal{P}$  or  $\mathcal{Q}$  would properly contain  $K_a$  or  $K_b$ , which yields a contradiction to the condition (2) in the choice of  $\mathcal{K}$ . Since  $\mathcal{K}(\mathcal{P})$  and  $\mathcal{K}(\mathcal{Q})$  are disjoint, the sets  $S(\mathcal{P})$  and  $S(\mathcal{Q})$  are contained in different of the two arcs of  $C \setminus (K_a \cup K_b)$ . Since  $G$  is connected,  $\mathcal{P}$  and  $\mathcal{Q}$  have a common arc  $A$ . This arc  $A$  intersects  $S(\mathcal{P})$  as well as  $S(\mathcal{Q})$ , that is, it intersects both arcs of  $C \setminus (K_a \cup K_b)$ . Hence either  $K_a$  or  $K_b$  is properly contained in  $A$ , which again yields a contradiction to the condition (2) in the choice of  $\mathcal{K}$ . This concludes the proof of the assertion.

Again let  $\mathcal{P}$  be a chain in  $\mathcal{F}$  of largest order and, subject to this, such that  $\mathcal{K}(\mathcal{P})$  has minimum order. Let  $\mathcal{K}(\mathcal{P}) = \{K_{a+1}, \dots, K_{b-1}\}$ .

In view of the desired statement, we may assume that  $\mathcal{F}$  contains a chain  $\mathcal{Q}$  of largest order such that  $K_{a+1}, K_{b-1} \notin \mathcal{K}(\mathcal{Q})$ . Among all such chains, we assume that  $\mathcal{Q}$  is chosen such that  $\mathcal{K}(\mathcal{Q})$  has minimum order. Let  $\mathcal{K}(\mathcal{Q}) = \{K_{\ell+1}, \dots, K_{m-1}\}$ . By the choice of  $\mathcal{P}$ , the sets  $\mathcal{K}(\mathcal{P})$  and  $\mathcal{K}(\mathcal{Q})$  are disjoint. By the assertion, we may assume that  $b = \ell + 1$ .

In view of the desired statement, we may assume that  $\mathcal{F}$  contains a chain  $\mathcal{R}$  of largest order such that  $K_{a+1}, K_{b-1}, K_{m-1} \notin \mathcal{K}(\mathcal{R})$ . Among all such chains, we assume that  $\mathcal{R}$  is chosen such that  $\mathcal{K}(\mathcal{R})$  has minimum order. Let  $\mathcal{K}(\mathcal{R}) = \{K_{p+1}, \dots, K_{q-1}\}$ . By the choice of  $\mathcal{P}$  and  $\mathcal{Q}$ , the sets  $\mathcal{K}(\mathcal{P}) \cup \mathcal{K}(\mathcal{Q})$  and  $\mathcal{K}(\mathcal{R})$  are disjoint. Applying the assertion to  $\mathcal{P}$  and  $\mathcal{R}$  as well as to  $\mathcal{Q}$  and  $\mathcal{R}$ , we obtain  $p = m - 1$  and  $q = a + 1$ , that is,  $\mathcal{K}(\mathcal{P}) \cup \mathcal{K}(\mathcal{Q}) \cup \mathcal{K}(\mathcal{R})$  is a partition of  $\mathcal{K}$ .

In view of the desired statement, we may assume that  $\mathcal{F}$  contains a chain  $\mathcal{S}$  of largest order such that  $K_{a+1}, K_{\ell+1}, K_{p+1} \notin \mathcal{K}(\mathcal{S})$ . We deduce from the choice of  $\mathcal{P}$  that  $\mathcal{K}(\mathcal{S})$  has at least as many elements as  $\mathcal{K}(\mathcal{P})$ . This implies that  $\mathcal{K}(\mathcal{S})$  is disjoint from  $\mathcal{K}(\mathcal{P})$ . Now, by the choice of  $\mathcal{Q}$ , this implies that  $\mathcal{K}(\mathcal{S})$  has at least as many elements as  $\mathcal{K}(\mathcal{Q})$ . This in turn implies that the set  $\mathcal{K}(\mathcal{S})$  is disjoint from  $\mathcal{K}(\mathcal{P}) \cup \mathcal{K}(\mathcal{Q})$ . Finally, by the choice of  $\mathcal{R}$ , this implies that  $\mathcal{K}(\mathcal{S})$  has at least as many elements as  $\mathcal{K}(\mathcal{R})$ . This in turn implies that the set  $\mathcal{K}(\mathcal{S})$  equals  $\mathcal{K}(\mathcal{R})$ , that is,  $\mathcal{K}(\mathcal{S})$  contains  $K_{p+1}$ , which is a contradiction.

This completes the proof that  $\text{lpt}(G)$  is at most 3.

From now on we assume that  $G$  is a 2-connected circular arc graph, that is, every two longest cycles in  $G$  — closed chains of largest order in  $\mathcal{F}$  — intersect. It is straightforward to see that the assertion also applies to closed chains instead of chains. Arguing exactly as above for closed chains in  $\mathcal{F}$  instead of chains in  $\mathcal{F}$  implies that  $\text{lct}(G) \leq 3$ .  $\square$

### Acknowledgement

This research was initiated during a visit of the first author to LORIA. It was supported by the French *Agence nationale de la recherche* under reference ANR 10 JCJC 0204 01.

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